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Dynamical parasuperalgebras of parasupersymmetric harmonic oscillator, cyclotron motion and Morse Hamiltonians

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Abstract. Simple quantum mechanical systems that have natural generalisations of superalgebras as dynamical algebras are discussed. We propose to call these new algebras parasuperalgebras. The symmetry generators of 'spin-1' particles in one-dimensional harmonic potentials are shown to realise an algebra endowed with a symmetric trilinear product. Spin-1 particles in constant magnetic fields or in two-dimensional harmonic potentials have analogous constants of motion that are described. It is also indicated how these conserved charges provide a spectrum-generating algebra for the parasupersymmetric generalisation of the Morse Hamiltonian. These concrete models should be useful in identifying the abstract properties of parasuperalgebras.

1. Introduction

Symmetries play an important role in theoretical physics, hence the interest in the mathematical structures that describe them. An example of such structures is that of Lie superalgebras, which are found to be realised by supersymmetry generators. We shall describe here certain one- and two-dimensional quantum harmonic oscillators, as well as other related systems, that have natural generalisations of superalgebras as dynamical symmetry algebras. These concrete models should be useful in identifying the abstract properties of those new algebras which we propose to call parasuperalgebras.

Consider the so-called fermionic creation and annihilation operators f and $f^\dagger$ that satisfy

$$\{f, f^\dagger\} \equiv ff^\dagger + f^\dagger f = 1 \qquad f^2 = f^{\dagger 2} = 0. \tag{1.1}$$

They are represented by the 2×2 matrices

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad f^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.2}$$

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Let the operators Q and $Q^\dagger$ be given by

$$Q = (p + \mathrm{i}W)f \qquad Q^\dagger = (p - \mathrm{i}W)f^\dagger \tag{1.3}$$

with $p = -\mathrm{i}\partial/\partial x$ and W an arbitrary real function of x. These supercharges provide a realisation of the following superalgebra:

$$\{Q, Q^\dagger\} = 2H \qquad Q^2 = {Q^\dagger}^2 = 0 \qquad [H, Q] = [H, Q^\dagger] = 0 \tag{1.4}$$

with

$$H = \tfrac{1}{2}\left(p^2 + W^2 + W'\sigma_3\right) \qquad \sigma_3 = [f^\dagger, f]. \tag{1.5}$$

(The prime stands for differentation.) In terms of the Hermitian charges

$$Q_1 = \frac{1}{\sqrt{2}}(Q + Q^\dagger) \qquad Q_2 = \frac{\mathrm{i}}{\sqrt{2}}(Q - Q^\dagger) \tag{1.6}$$

one has equivalently

$$\{Q_i, Q_j\} = 2\delta_{ij}H \qquad [H, Q_i] = 0 \qquad i, j = 1, 2. \tag{1.7}$$

The operator H is interpreted as the Hamiltonian of a 'spin-$\frac{1}{2}$' particle moving in the potential $\frac{1}{2}W^2$ and in the magnetic field $\frac{1}{2}W'$ directed along the 3-axis. Such a Hamiltonian is said to be supersymmetric [1].

The parasupersymmetric generalisation of the Hamiltonians (1.5) is obtained by replacing in the above construction the canonical fermionic creation and annihilation operators $f^\dagger$ and f by their parafermionic counterparts $a^\dagger$ and a of order 2. These satisfy [2]

$$a^2a^\dagger + a^\dagger a^2 = 2a \qquad aa^\dagger a = 2a \qquad a^3 = 0 \tag{1.8}$$

and are represented by the following 3×3 matrices:

$$a = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad a^\dagger = \sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{1.9}$$

Consider now [3] a new set of operators Q ($Q^\dagger$) and H given by

$$Q = \frac{1}{2\sqrt{2}}\left[(p + \mathrm{i}W_1)a^\dagger a^2 + (p + \mathrm{i}W_2)a^2a^\dagger\right] \tag{1.10}$$

$$H = \tfrac{1}{2}\left[p^2 + \tfrac{1}{2}(W_1^2 + W_2^2) + \tfrac{3}{2}(W_1' - W_2') + W_2'a^\dagger a - W_1'aa^\dagger\right]. \tag{1.11}$$

Provided the functions $W_1(x)$ and $W_2(x)$ satisfy

$$(W_2^2 - W_1^2)' + (W_2 + W_1)'' = 0 \tag{1.12}$$

one finds that the following parasuperalgebra is realised:

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 4QH \tag{1.13a}$$

$$Q^{\dagger 2}Q + Q^\dagger QQ^\dagger + QQ^{\dagger 2} = 4Q^\dagger H \tag{1.13b}$$

$$Q^3 = Q^{\dagger 3} = 0 \tag{1.13c}$$

$$[H, Q] = [H, Q^\dagger] = 0. \tag{1.13d}$$

If one uses instead the Hermitian operators

$$Q_1 = \tfrac{1}{2}(Q + Q^\dagger) \qquad Q_2 = \tfrac{1}{2}\mathrm{i}(Q - Q^\dagger) \tag{1.14}$$

the above equations can be rewritten in the form

$$Q_i(\{Q_j, Q_k\} - 2\delta_{jk}H) + Q_j(\{Q_k, Q_i\} - 2\delta_{ki}H) + Q_k(\{Q_i, Q_j\} - 2\delta_{ij}H) = 0 \tag{1.15a}$$

$$[H, Q_i] = 0 \qquad (i, j, k = 1, 2). \tag{1.15b}$$

The relations (1.13) or (1.15) characterise the basic second-order parasuperalgebra. The trilinear product rule for the fermionic elements represents its distinctive feature. (Parasuperalgebras of order p would involve a $(p + 1)$-linear product.) When real charges are used, we immediately note that the expressions defining the anticommutation relations of the corresponding superalgebra factor out.

For particular values of W_1 and W_2, the parasupersymmetric H can have further symmetries. There are then constants of motion (in addition to the charges Q and $Q^\dagger$) that enlarge the parasuperalgebra (1.13). For example, when $W_1 = \lambda/x$ and $W_2 = (\lambda + 1)/x$, the corresponding Hamiltonian is found to admit a dynamical second-order parasuperalgebra that generalises the OSp(2, 1) conformal superalgebra [4]. Often, these higher symmetries allow for an algebraic resolution of the dynamics; this is the case for the example just quoted.

We shall here analyse in this respect, Hamiltonians of the harmonic oscillator type. In section 2, we describe the second-order dynamical parasuperalgebra of the one-dimensional harmonic oscillator. Among the symmetry generators one has, of course, those of the ordinary bosonic oscillator that span a subalgebra isomorphic to the invariance algebra of a free non-relativistic spinless particle, known as the Schrödinger algebra. In addition, there are twelve fermionic charges and four more bosonic constants. Together, all these conserved quantities form a closed algebraic set. In section 3, we focus on two-dimensional systems. We first give the constants of motion for a spin-1 particle in a constant magnetic field and then establish the relation between this system and the two-dimensional harmonic oscillator with one parafermionic degree of freedom. Finally, in section 4, we indicate how the symmetry generators of the latter provide a spectrum-generating parasuperalgebra for the parasupersymmetric Morse Hamiltonian.

2. Dynamical parasuperalgebra of the one-dimensional parasupersymmetric harmonic oscillator

We shall be interested in situations where the functions W_1 and W_2 entering in (1.10) and (1.11) satisfy

$$W_1' = W_2'. \tag{2.1}$$

When this is so, the Hamiltonian (1.11) can be written as

$$H = \tfrac{1}{2}p^2 + V(x) + \Sigma_3 B(x) \tag{2.2}$$

with

$$V(x) = \tfrac{1}{4}(W_1^2 + W_2^2) \qquad B(x) = W_1' = W_2' \tag{2.3}$$

and

$$\Sigma_3 = \tfrac{1}{2}[a^\dagger, a] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2.4}$$

This operator H is of the same form as the Hamiltonian (1.5) except for the fact that σ_3 has been replaced by Σ_3; it governs the one-dimensional motion of a 'spin-1' particle in a potential $V(x)$ and a magnetic field $B(x)$ directed along the third axis.

Conditions (1.12) and (2.1) possess only two simultaneous solutions [3]; they are

$$W_1 = \omega x + \beta \qquad W_2 = W_1 \tag{2.5}$$

and

$$W_1 = \omega \mathrm{e}^{-\lambda x} + \beta \qquad W_2 = W_1 + \lambda \tag{2.6}$$

with ω, β and λ arbitrary constants. The first, (2.5), leads to a potential V which is harmonic and a magnetic field B which is constant; the second, (2.6), yields for V a Morse potential and for B an exponential field.

The parasupersymmetric Morse system will be considered in the final section. First, we shall study the symmetries of the one-dimensional parasupersymmetric harmonic oscillator associated with the solution (2.5) and for which the Hamiltonian and conserved parasupercharge are

$$H = \tfrac{1}{2}(p^2 + \omega^2 x^2 + 2\beta\omega x + \beta^2) + \omega\Sigma_3 \tag{2.7}$$

$$Q = \frac{1}{\sqrt{2}}[p + \mathrm{i}(\omega x + \beta)]\,a. \tag{2.8}$$

It will be convenient to use the following bosonic annihilation and creation operators:

$$\alpha = \frac{1}{\sqrt{2\omega}}(p - \mathrm{i}W) \qquad \alpha^\dagger = \frac{1}{\sqrt{2\omega}}(p + \mathrm{i}W) \qquad [\alpha, \alpha^\dagger] = 1 \tag{2.9}$$

with $W = \omega x + \beta$. In terms of α and $\alpha^\dagger$, the Hamiltonian (2.7) can be re-expressed as

$$H = H_0 + \omega\Sigma_3 \tag{2.10}$$

where

$$H_0 = \tfrac{1}{2}\omega\{\alpha, \alpha^\dagger\}. \tag{2.11}$$

With H determining the time evolution of the dynamical variables, it is immediately possible to check that the following quantities:

$$X_{k,l,m,n} = \exp[-\mathrm{i}\omega(k-l+m-n)t](\alpha^\dagger)^k \alpha^l (a^\dagger)^m a^n \qquad k,l,m,n \in \mathcal{N} \tag{2.12}$$

are conserved, in other words that they satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{k,l,m,n} = \frac{\partial}{\partial t} X_{k,l,m,n} + \mathrm{i}[H, X_{k,l,m,n}] = 0. \tag{2.13}$$

We now want to determine the maximal, closed and finite-dimensional algebraic set that can be formed out of these constants of motion. Owing to the properties of the fermionic annihilation and creation operators, this set is expected to have the structure of a second-order parasuperalgebra.

Let us first observe that the algebra of real 3×3 matrices can be endowed with a $\mathcal{Z}_2$ grading. It is easy to check that the following nine matrices form a basis for $\mathrm{gl}(3, \mathbb{R})$:

$$\text{odd elements:} \qquad a,\ a^\dagger,\ b = \tfrac{1}{2}(a^\dagger a^2 - a^2 a^\dagger),\ b^\dagger \tag{2.14a}$$

$$\text{even elements:} \qquad I,\ a^2,\ {a^\dagger}^2,\ \Sigma_3 = \tfrac{1}{2}[a^\dagger, a],\ Y = \tfrac{1}{2}\{a^\dagger, a\} \tag{2.14b}$$

where I is the identity operator. They have been grouped according to their grading, which is unambiguously determined since a and $a^\dagger$ are by definition fermionic. (We may remark that $b^2 = -a^2$, $\Sigma_3 = \frac{1}{2}[b^\dagger, b]$, $Y = \frac{1}{2}\{b^\dagger, b\}$.) The even elements must by themselves form a Lie algebra under ordinary commmutation and indeed, this algebra is immediately found to be $\mathrm{SU}(2) \oplus \mathrm{U}(1) \oplus \mathbb{R}$.

The bosonic sector of the invariance algebra of our parasupersymmetric harmonic oscillator Hamiltonian H will of course possess as subalgebra, the invariance algebra of the ordinary bosonic harmonic oscillator Hamiltonian H_0. This algebra is known as the one-dimensional Schrödinger algebra [5]; it is isomorphic to the invariance algebra of the free bosonic Hamiltonian and is denoted by Sch(1). It is generated by those constants $X_{k,l,m,n}$ which are of first and second order in α and $\alpha^\dagger$; i.e. H_0, I and

$$P = \sqrt{\omega}\mathrm{e}^{-\mathrm{i}\omega t}\alpha^\dagger \qquad P^\dagger = \sqrt{\omega}\mathrm{e}^{\mathrm{i}\omega t}\alpha \tag{2.15a}$$

$$A = \omega \mathrm{e}^{-2\mathrm{i}\omega t}{\alpha^\dagger}^2 \qquad A^\dagger = \omega \mathrm{e}^{2\mathrm{i}\omega t}\alpha^2. \tag{2.15b}$$

In order to get the complete bosonic subalgebra of the invariance algebra of H, we must still supplement the above generators with those constants, quadratic in the fermionic annihilation and creation operators, that correspond to the even sector of $\mathrm{gl}(3, \mathbb{R})$. This means adjoining

$$L_1 = \tfrac{1}{4}\left(\mathrm{e}^{2\mathrm{i}\omega t}a^2 + \mathrm{e}^{-2\mathrm{i}\omega t}{a^\dagger}^2\right) \tag{2.16a}$$

$$L_2 = \tfrac{1}{4}\mathrm{i}\left(\mathrm{e}^{2\mathrm{i}\omega t}a^2 - \mathrm{e}^{-2\mathrm{i}\omega t}{a^\dagger}^2\right) \tag{2.16b}$$

$$L_3 = \tfrac{1}{2}\Sigma_3 = \tfrac{1}{4}[a^\dagger, a] \tag{2.16c}$$

and Y, to the Sch(1) basis $\{H_0, P, P^\dagger, A, A^\dagger, I\}$. (The proper explicit time dependences have been introduced so that all generators are conserved.) The full bosonic invariance algebra is then identified as being $\mathrm{Sch}(1)\oplus\mathrm{SU}(2)\oplus\mathrm{U}(1)$. The commutation relations are easily computed and the non vanishing commutators are given by:

$$[H_0,A]=2\omega A \qquad [H_0,A^\dagger]=-2\omega A^\dagger \qquad [A^\dagger,A]=4\omega H_0 \tag{2.17a}$$

$$\begin{array}{ll}[H_0,P]=\omega P & [H_0,P^\dagger]=-\omega P^\dagger \\ [A^\dagger,P]=2\omega P^\dagger & [A,P^\dagger]=-2\omega P\end{array} \tag{2.17b}$$

$$[P^\dagger,P]=\omega I \tag{2.17c}$$

$$[L^i,L^j]=i\varepsilon_{ijk}L^k \qquad i,j,k=1,2,3. \tag{2.17d}$$

The fermionic generators must necessarily involve one of the four odd basis elements of $\mathrm{gl}(3,\mathbb{R})$, and closure requires them to be at most linear in the bosonic creation and annihilation operators. (This last point will be obvious when we present the fermionic structure relations.) It will be convenient to organise these fermionic charges in irreducible multiplets with respect to the SU(2) generated by L_i, $i=1,2,3$. In this respect, one notes that both

$$T_\mu=\begin{pmatrix} a\,\mathrm{e}^{\mathrm{i}\omega t} \\ -b^\dagger \mathrm{e}^{-\mathrm{i}\omega t}\end{pmatrix}_\mu \qquad \bar{T}_\mu=\begin{pmatrix} b\,\mathrm{e}^{\mathrm{i}\omega t} \\ a^\dagger \mathrm{e}^{-\mathrm{i}\omega t}\end{pmatrix}_\mu=(-\mathrm{i}\sigma_2 T^*)_\mu \tag{2.18}$$

transform as 2-spinors under this SU(2). Indeed

$$[L^i,T_\mu]=-\tfrac{1}{2}\sigma^i_{\mu\nu}T_\nu \qquad [L^i,\bar{T}_\mu]=-\tfrac{1}{2}\sigma^i_{\mu\nu}\bar{T}_\nu \qquad \mu,\nu=1,2. \tag{2.19}$$

One also observes that

$$[Y,T_\mu]=\bar{T}_\mu \qquad [Y,\bar{T}_\mu]=T_\mu. \tag{2.20}$$

It follows that all the other admissible fermionic operators can themselves be cast as 2-spinors. In addition to T_μ and $\bar{T}_\mu$ one has

$$\begin{array}{ll} Q_\mu=PT_\mu & \bar{Q}_\mu=P\bar{T}_\mu \\ S_\mu=P^\dagger T_\mu & \bar{S}_\mu=P^\dagger\bar{T}_\mu. \end{array} \tag{2.21a}$$

The transformations properties of these constants under the bosonic symmetry operations are given by

$$[H_0,Q_\mu]=\omega Q_\mu \qquad [H_0,S_\mu]=-\omega S_\mu \qquad [H_0,T_\mu]=0 \tag{2.22a}$$

$$[A,Q_\mu]=0 \qquad [A,S_\mu]=-2\omega Q_\mu \qquad [A,T_\mu]=0 \tag{2.22b}$$

$$[A^\dagger,Q_\mu]=2\omega S_\mu \qquad [A^\dagger,S_\mu]=0 \qquad [A^\dagger,T_\mu]=0 \tag{2.22c}$$

$$[P,Q_\mu]=0 \qquad [P,S_\mu]=-\omega T_\mu \qquad [P,T_\mu]=0 \tag{2.22d}$$

$$[P^\dagger,Q_\mu]=\omega T_\mu \qquad [P^\dagger,S_\mu]=0 \qquad [P^\dagger,T_\mu]=0 \tag{2.22e}$$

$$[Y,Q_\mu]=\bar{Q}_\mu \qquad [Y,S_\mu]=\bar{S}_\mu \qquad [Y,T_\mu]=\bar{T}_\mu \tag{2.22f}$$

$$[L^i,Q_\mu]=-\tfrac{1}{2}\sigma^i_{\mu\nu}Q_\nu \qquad [L^i,S_\mu]=-\tfrac{1}{2}\sigma^i_{\mu\nu}S_\nu \qquad [L^i,T_\mu]=-\tfrac{1}{2}\sigma^i_{\mu\nu}T_\nu. \tag{2.22g}$$

The commutation relations involving $\bar{Q}$, $\bar{S}$ and $\bar{T}$ are obtained from the above formulae by effecting the substitutions $Q\leftrightarrow\bar{Q}$, $S\leftrightarrow\bar{S}$, $T\leftrightarrow\bar{T}$.

There now only remains to give the products of the fermionic basis elements. Let us introduce the symbol $\{A_1, A_2, \ldots, A_n\}$ to represent the n-linear symmetric product of $A_1, A_2, \ldots, A_n$:

$$\{A_1, A_2, \ldots, A_n\} = \sum_{\sigma \in S_n} A_{\sigma(1)} A_{\sigma(2)} \cdots A_{\sigma(n)}, \tag{2.23}$$

with S_n the symmetric group of n objects. In particular we have

$$\{A, B, C\} = A\{B, C\} + B\{C, A\} + C\{A, B\}. \tag{2.24}$$

Since we are using parafermionic variables of second order and since these satisfy cubic identities, we expect that (2.24) is the right product to take between odd elements in order to define our algebra†. We shall now complete the list of the structure relations of the dynamical parasuperalgebra of the parasupersymmetric harmonic oscillator by stating what all these trilinear products of the parafermionic charges are equal to. In what follows $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\mu, \nu, \rho = 1, 2$:

$$\begin{aligned}
\{Q_\mu, Q_\nu, \bar{Q}_\rho\} &= 4\epsilon_{\mu\rho}[AQ]_\nu + (\mu \leftrightarrow \nu) \\
\{S_\mu, S_\nu, \bar{S}_\rho\} &= 4\epsilon_{\mu\rho}[A^\dagger S]_\nu + (\mu \leftrightarrow \nu) \\
\{T_\mu, T_\nu, \bar{T}_\rho\} &= 4\epsilon_{\mu\rho}[IT]_\nu + (\mu \leftrightarrow \nu)
\end{aligned} \tag{2.25a}$$

$$\begin{aligned}
\{Q_\mu, Q_\nu, \bar{S}_\rho\} &= 4\epsilon_{\mu\rho}[(H_0 + \tfrac{2}{3}\omega L^i \sigma^i)Q]_\nu + (\mu \leftrightarrow \nu) \\
\{S_\mu, S_\nu, \bar{Q}_\rho\} &= 4\epsilon_{\mu\rho}[(H_0 - \tfrac{2}{3}\omega L^i \sigma^i)S]_\nu + (\mu \leftrightarrow \nu) \\
\{Q_\mu, Q_\nu, \bar{T}_\rho\} &= 4\epsilon_{\mu\rho}[PQ]_\nu + (\mu \leftrightarrow \nu) \\
\{T_\mu, T_\nu, \bar{Q}_\rho\} &= 4\epsilon_{\mu\rho}[PT]_\nu + (\mu \leftrightarrow \nu) \\
\{S_\mu, S_\nu, \bar{T}_\rho\} &= 4\epsilon_{\mu\rho}[P^\dagger S]_\nu + (\mu \leftrightarrow \nu) \\
\{T_\mu, T_\nu, \bar{S}_\rho\} &= 4\epsilon_{\mu\rho}[P^\dagger T]_\nu + (\mu \leftrightarrow \nu)
\end{aligned} \tag{2.25b}$$

$$\begin{aligned}
\{Q_\mu, S_\nu, \bar{Q}_\rho\} &= 2\epsilon_{\mu\rho}[AS + (H_0 - \tfrac{2}{3}\omega L^i \sigma^i)Q]_\nu + (\mu \leftrightarrow \nu) - \omega\epsilon_{\mu\nu}\bar{Q}_\rho \\
\{Q_\mu, S_\nu, \bar{S}_\rho\} &= 2\epsilon_{\mu\rho}[A^\dagger Q + (H_0 + \tfrac{2}{3}\omega L^i \sigma^i)S]_\nu + (\mu \leftrightarrow \nu) - \omega\epsilon_{\mu\nu}\bar{S}_\rho \\
\{Q_\mu, T_\nu, \bar{Q}_\rho\} &= 2\epsilon_{\mu\rho}[AT + PQ]_\nu + (\mu \leftrightarrow \nu) \\
\{Q_\mu, T_\nu, \bar{T}_\rho\} &= 2\epsilon_{\mu\rho}[PT + IQ]_\nu + (\mu \leftrightarrow \nu) \\
\{S_\mu, T_\nu, \bar{S}_\rho\} &= 2\epsilon_{\mu\rho}[A^\dagger T + P^\dagger S]_\nu + (\mu \leftrightarrow \nu) \\
\{S_\mu, T_\nu, \bar{T}_\rho\} &= 2\epsilon_{\mu\rho}[P^\dagger T + IS]_\nu + (\mu \leftrightarrow \nu)
\end{aligned} \tag{2.25c}$$

† This product has already been used in the definition of second-order parasupersymmetric quantum mechanics.

$$
\begin{aligned}
\{Q_\mu, S_\nu, \bar{T}_\rho\} &= 2\epsilon_{\mu\rho}[P^\dagger Q + PS]_\nu + (\mu \leftrightarrow \nu) \\
\{S_\mu, T_\nu, \bar{Q}_\rho\} &= 2\epsilon_{\mu\rho}[PS + (H_0 - \tfrac{2}{3}\omega L^i\sigma^i)T]_\nu + (\mu \leftrightarrow \nu) + \omega\epsilon_{\mu\nu}\bar{T}_\rho \\
\{T_\mu, Q_\nu, \bar{S}_\rho\} &= 2\epsilon_{\mu\rho}[P^\dagger Q + (H_0 + \tfrac{2}{3}\omega L^i\sigma^i)T]_\nu + (\mu \leftrightarrow \nu) + \omega\epsilon_{\mu\nu}\bar{T}_\rho
\end{aligned}
\qquad (2.25d)
$$

$$
\begin{aligned}
&\{Q_\mu, Q_\nu, S_\rho\} = 2\omega\epsilon_{\mu\rho}Q_\nu + (\mu \leftrightarrow \nu) && \{Q_\mu, Q_\nu, Q_\rho\} = 0 \\
&\{S_\mu, S_\nu, Q_\rho\} = -2\omega\epsilon_{\mu\rho}S_\nu + (\mu \leftrightarrow \nu) && \{S_\mu, S_\nu, S_\rho\} = 0 \\
&\{Q_\mu, Q_\nu, T_\rho\} = 0 && \{T_\mu, T_\nu, T_\rho\} = 0 \\
&\{T_\mu, T_\nu, Q_\rho\} = 0 && \\
&\{S_\mu, S_\nu, T_\rho\} = 0 && \{Q_\mu, S_\nu, T_\rho\} = 2\omega\epsilon_{\mu\nu}T_\rho . \\
&\{T_\mu, T_\nu, S_\rho\} = 0 &&
\end{aligned}
\qquad (2.25e)
$$

The relations of the form $\{A, \bar{B}, \bar{C}\}$ or $\{\bar{A}, \bar{B}, \bar{C}\}$ can be obtained from the above using

$$\{\bar{A}, \bar{B}, \bar{C}\} = -\overline{\{A, B, C\}} \qquad (2.26a)$$

$$\{\bar{A}, \bar{B}, C\} = -\overline{\{A, B, \bar{C}\}}. \qquad (2.26b)$$

These identities are easily proven with the help of (2.22f).

In summary, the one-dimensional parasupersymmetric harmonic oscillator has been found to admit a 22-dimensional dynamical parasuperalgebra. The operators $H_0, Y, P, P^\dagger, A, A^\dagger, I$ and L_i, $i = 1, 2, 3$ defined in (2.11), (2.14b), (2.15) and (2.16) provide a basis for its bosonic subalgebra while the charges $Q_\mu, \bar{Q}_\mu, S_\mu, \bar{S}_\mu$, T_μ and $\bar{T}_\mu$, $\mu = 1, 2$, introduced in (2.18) and (2.21), span its parafermionic sector. The structure relations are given in (2.17), (2.22) and (2.25). This second-order parasuperalgebra generalises the dynamical superalgebra (described in [6]) of the supersymmetric one-dimensional harmonic oscillator.

3. Parasupersymmetries of spin-1 particles in cyclotron motion

We shall now draw from the preceding section to describe the parasupersymmetries of spin-1 particles moving in a constant magnetic field B (taken along the 3-direction). The Hamiltonian that governs the dynamics in the plane perpendicular to the magnetic field is given by

$$H = \tfrac{1}{2}\left(\pi_1^2 + \pi_2^2\right) + B\Sigma_3 \qquad (3.1)$$

with $\pi_i = p_i - A_i$ and $[\pi_i, \pi_j] = \mathrm{i}\varepsilon_{ij}B$, $i, j = 1, 2$. We have taken the charges of the particles to be one.

Observe now that the following correspondence:

$$\pi_1 \leftrightarrow p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} \qquad \pi_2 \leftrightarrow -(Bx + \beta) \qquad (3.2)$$

can be effected consistently as $[\pi_1, \pi_2]$ and $[p, -(Bx+\beta)]$ are both equal to $\mathrm{i}B$. Under these substitutions H gets transformed into

$$H = \tfrac{1}{2}(p^2 + B^2x^2 + 2\beta Bx + \beta^2) + B\Sigma_3 \tag{3.3}$$

an expression which is immediately recognised as the Hamiltonian (2.7) of our parasupersymmetric one-dimensional harmonic oscillator (with ω replaced by B). From the parasupercharge $(1/\sqrt{2})[p - \mathrm{i}(Bx+\beta)]a$ given in (2.8) and associated with the above harmonic oscillator Hamiltonian, we may now simply obtain a parasupercharge for the cyclotron motion Hamiltonian (3.1) by using (3.2) in the reverse direction. The parasuperalgebra (1.13) will thus also be realised with H given as in (3.1) and with

$$Q = \frac{1}{\sqrt{2}}(\pi_1 + \mathrm{i}\pi_2)\,a. \tag{3.4}$$

This can of course be checked directly.

It is, moreover, clear that all the other conserved charges and parasupercharges of the one-dimensional harmonic oscillator can similarly be converted into constants of our cyclotron motion. This, however, does not provide all the symmetry generators of the Hamiltonian (3.1). For instance, space translations combined with gauge transformations leave H invariant. This entails the conservation of the so-called Lippmann–Johnson [7] constants

$$C_1 = \pi_1 - Bx_2 \qquad C_2 = \pi_2 + Bx_1 \tag{3.5}$$

which satisfy

$$[C_i, C_j] = -\mathrm{i}\varepsilon_{ij}B \qquad [C_i, \pi_j] = 0 \qquad i,j = 1,2 \tag{3.6}$$

and hence $[C_i, H] = 0$.

If one defines

$$\pi_\pm = \frac{1}{\sqrt{2}}(\pi_1 \pm \mathrm{i}\pi_2) \qquad \pi_+^\dagger = \pi_- \tag{3.7}$$

and similarly

$$C_\pm = \frac{1}{\sqrt{2}}(C_1 \pm \mathrm{i}C_2) \tag{3.8}$$

one has

$$[\pi_+, \pi_-] = -[C_+, C_-] = B \tag{3.9a}$$

$$H = \tfrac{1}{2}\{\pi_+, \pi_-\} + B\Sigma_3 \tag{3.9b}$$

$$Q = \pi_+ a. \tag{3.9c}$$

Then, the following quantities

$$\Pi_\pm(t) = \pi_\pm \mathrm{e}^{\pm\mathrm{i}Bt} \qquad C_\pm \qquad T_\mu = \begin{pmatrix} a\,\mathrm{e}^{\mathrm{i}Bt} \\ -b^\dagger \mathrm{e}^{-\mathrm{i}Bt} \end{pmatrix}_\mu \qquad \bar{T}_\mu = \begin{pmatrix} b\,\mathrm{e}^{\mathrm{i}Bt} \\ a^\dagger \mathrm{e}^{-\mathrm{i}Bt} \end{pmatrix}_\mu \tag{3.10}$$

are easily seen to have vanishing total time derivative when the time evolution is controlled by H.

It turns out that a basis for the maximal finite-dimensional dynamical parasuperalgebra of this problem can be obtained by adding to the set $\{\Pi_\pm, C_\pm, T_\mu, \bar{T}_\mu\}$ all quadratic monomials that can be formed out of its elements. This assertion is best inferred by observing that the two-dimensional harmonic oscillator whose Hamiltonian is

$$H_{\text{ho}} = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + \tfrac{1}{2}\left(\tfrac{1}{2}B\right)^2\left(x_1^2 + x_2^2\right) + \tfrac{1}{2}B\Sigma_3 \tag{3.11}$$

and the cyclotron problem possess isomorphic dynamical parasuperalgebras.

Let us choose a gauge in which

$$\boldsymbol{A} = \left(-\tfrac{1}{2}Bx_2, \tfrac{1}{2}Bx_1\right). \tag{3.12}$$

In this case

$$\pi_+ = \sqrt{2B}\,\alpha_+ \qquad \pi_- = \sqrt{2B}\,\alpha_+^\dagger \tag{3.13a}$$
$$C_+ = \sqrt{2B}\,\alpha_-^\dagger \qquad C_- = \sqrt{2B}\,\alpha_- \tag{3.13b}$$

with

$$\alpha_\pm = \tfrac{1}{2}(\alpha_1 \pm i\alpha_2) \tag{3.14a}$$

and

$$\alpha_i = \frac{1}{\sqrt{B}}\left(p_i - \mathrm{i}\tfrac{1}{2}Bx_i\right) \qquad [\alpha_i, \alpha_j^\dagger] = \delta_{ij} \qquad i,j = 1,2. \tag{3.14b}$$

Since H_{ho} can be written as

$$H_{\text{ho}} = \tfrac{1}{2}B\left(\{\alpha_+, \alpha_+^\dagger\} + \{\alpha_-, \alpha_-^\dagger\} + \Sigma_3\right) \tag{3.15}$$

we see that

$$H = H_{\text{ho}} + \tfrac{1}{2}B\left(\{\alpha_+, \alpha_+^\dagger\} - \{\alpha_-, \alpha_-^\dagger\}\right) + \tfrac{1}{2}B\Sigma_3. \tag{3.16}$$

At $t = 0$, the dynamical algebra of H_{ho} consists, as we know, of all quadratic polynomials in $\alpha_\pm$, $\alpha_\pm^\dagger$ (or equivalently $\pi_\pm$, $C_\pm$), $T_\mu(t=0)$ and $\bar{T}_\mu(t=0)$. From (3.16), we see that H is actually an element of this algebra; it therefore shares with H_{ho} the same dynamical algebra. The only difference will lie in the explicit time dependence of the conserved generators if one chooses H_{ho} instead of H as the generator of time translations. It is a simple exercise to figure out these dependences. In fact, note that $\{\alpha_-, \alpha_-^\dagger\} - \{\alpha_+, \alpha_+^\dagger\} = x_1p_2 - x_2p_1$; we thus have

$$H = H_{\text{ho}} - \tfrac{1}{2}B(2M_2 - \Sigma_3) \qquad 2M_2 = x_1p_2 - x_2p_1. \tag{3.17}$$

It follows that quantities $X(t) = \mathrm{e}^{-\mathrm{i}Ht}X(0)\mathrm{e}^{\mathrm{i}Ht}$, conserved under H, will be converted into quantities $X_{\text{ho}}(t) = \exp[-\mathrm{i}H_{\text{ho}}t]X(0)\exp[\mathrm{i}H_{\text{ho}}t]$, conserved under H_{ho} by a time-dependent spatial rotation:

$$X(t) = \exp[\mathrm{i}B(M_2 - \Sigma_3/2)t]X_{\text{ho}}(t)\exp[-\mathrm{i}B(M_2 - \Sigma_3/2)t]. \tag{3.18}$$

The structure relations of the second-order dynamical parasuperalgebra of (either one of) the systems discussed in this section, naturally extend those of the one-dimensional harmonic oscillator dynamical parasuperalgebra given in section 2. They can be found in [8] and will not be reproduced here.

4. Spectrum-generating algebra of the parasupersymmetric Morse Hamiltonian

The parasupersymmetric Morse Hamiltonian has already been encountered in section 2. Besides the harmonic oscillator, it is the only other parasupersymmetric Hamiltonian that can be written in the form $H = \frac{1}{2}p^2 + \frac{1}{4}(W_1^2 + W_2^2) + \Sigma_3 W_1'$ with $W_1' = W_2'$. It is obtained by substituting

$$W_1 = -\tfrac{1}{2}h\mathrm{e}^{-x} + \tfrac{1}{2}(h-1) \qquad W_2 = W_1 + 1 \tag{4.1}$$

(see (2.6)) in H, one thus gets:

$$2H_{\mathrm{M}} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tfrac{1}{4}h^2(\mathrm{e}^{-x} - 1)^2 + h\mathrm{e}^{-x}\Sigma_3 + \tfrac{1}{4}. \tag{4.2}$$

It is known that the dynamics of the bosonic [9] and supersymmetric [10] (one-dimensional) Morse problems is amenable to an algebraic treatment. In fact the bound states of the bosonic, respectively supersymmetric, Morse Hamiltonians are in correspondence with the basis vectors of Sp(1), respectively OSp(1,1), unitary representations. We shall here show that the spectrum of the parasupersymmetric Hamiltonian H_{M} is similarly related to representations of the dynamical parasuperalgebra of the two-dimensional harmonic oscillator treated in the last section.

Let $N_0 = \alpha_1^\dagger\alpha_1 + \alpha_2^\dagger\alpha_2$. From now on we shall take $B = 2$. A set of labels for the quantum states of the Hamiltonian

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + \tfrac{1}{2}\left(x_1^2 + x_2^2\right) + \Sigma_3 \tag{4.3}$$

is provided by the following eigenvalue equations:

$$N_0|n_0, \delta, m_2\rangle = n_0|n_0, \delta, m_2\rangle \qquad n_0 = 0, 1, 2... \tag{4.4a}$$

$$\Sigma_3|n_0, \delta, m_2\rangle = \delta|n_0, \delta, m_2\rangle \qquad \delta = -1, 0, 1 \tag{4.4b}$$

$$M_2|n_0, \delta, m_2\rangle = m_2|n_0, \delta, m_2\rangle \qquad -\tfrac{1}{2}n_0 \leq m_2 \leq \tfrac{1}{2}n_0. \tag{4.4c}$$

In obtaining the range of values for m_2 we have used the fact that $M_1 = \frac{1}{2}(\alpha_1^\dagger\alpha_2 + \alpha_2^\dagger\alpha_1)$, $M_2 = \frac{1}{2}\mathrm{i}(\alpha_2^\dagger\alpha_1 - \alpha_1^\dagger\alpha_2)$ and $M_3 = \frac{1}{2}(\alpha_1^\dagger\alpha_1 - \alpha_2^\dagger\alpha_2)$ form an SU(2) algebra with

$$M^2 = M_1^2 + M_2^2 + M_3^2 = \tfrac{1}{4}N_0(N_0 + 2). \tag{4.5}$$

The equations (4.4) also define a basis for the representation space of the dynamical parasuperalgebra of H. We shall now indicate how they further provide the eigenfunctions of the parasupersymmetric Morse Hamiltonian H_{M}.

Instead of n_0, we can equivalently use as quantum number the eigenvalue $h = n_0 + 2\delta + 1$ of the operator

$$\tilde{H} = N_0 + 2\Sigma_3 + 1. \tag{4.6}$$

The basis states will thus be identified as $|h, \delta, m_2\rangle$ and (4.4a) replaced by

$$\tilde{H}|h, \delta, m_2\rangle = h|h, \delta, m_2\rangle \qquad h = -1, 0, 1, 2, \ldots. \tag{4.7}$$

In terms of the polar coordinates

$$x_1 = r\cos\phi \qquad x_2 = r\sin\phi \tag{4.8}$$

$M_2 = -\frac{1}{2}\mathrm{i}\partial/\partial\phi$. Equation (4.4*c*) is then immediately integrated and one gets

$$\langle r,\phi|h,\delta,m_2\rangle = \mathrm{e}^{2\mathrm{i}m_2\phi}\Psi_{h,\delta,m_2}(r). \tag{4.9}$$

Separating the variables in

$$\langle r,\phi|\tilde{H}|h,\delta,m_2\rangle = h\langle r,\phi|h,\delta,m_2\rangle \tag{4.10}$$

one finds

$$\left[\tfrac{1}{2}\left(-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}+\frac{4m_2^2}{r^2}+r^2\right)+2\delta\right]\Psi_{h,\delta,m_2}(r) = h\Psi_{h,\delta,m_2}(r). \tag{4.11}$$

The solutions to this equation can be obtained either directly or by applying ladder operators on the ground-state wavefunctions. They are given by

$$\Psi_{h,\delta,m_2}(r) = C_\delta r^{2m_2}\mathrm{e}^{-r/2}F(m_2-\tfrac{1}{2}(h-2\delta-1),2m_2+1;r^2) \tag{4.12}$$

with C_δ normalisation constants and F a confluent hypergeometric function.

The eigenfunctions of the Morse Hamiltonian can now be obtained from (4.12) by a mere change of variable. Indeed, setting $r^2 = h\mathrm{e}^{-x}$ in (4.11) and multiplying this equation on both sides by r^2 yields

$$H_{\mathrm{M}}\Psi_{h,\delta,m_2}(x) = \left[-\tfrac{1}{2}m_2^2+\tfrac{1}{8}(h^2+1)\right]\Psi_{h,\delta,m_2}(x) \tag{4.13}$$

with H_{M} as given in (4.2).

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